ON A GYROSCOPE MOUNTED IN A UNIVERSAL SUSPENSION [ON GIMBALS]

(O GIROSKOPE V KARDANOVOM PODVESE)

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Consider a symmetric heavy gyroscope rotating in a universal suspension with the axis of the external ring vertical. The resulting motion has much in common with the well-studied case of Lagrange motion. One might also consider the simpler question of stability with respect to the angle of nutation [1].

Questions concerning the stability for all variables of the problem have been recently dealt with by Magnus [2] and Rumiantsev [3].

1. Let us consider a gyroscope in a universal suspension as shown in Fig. 1. We denote by x_1, y_1, and z_1 the fixed coordinate system, by ψ the angle of rotation of the (external) ring, by θ the angle of rotation of the housing in the ring, by x, y, and z axes fixed on the housing, and by ϕ the angle of rotation of the gyroscope in the housing.

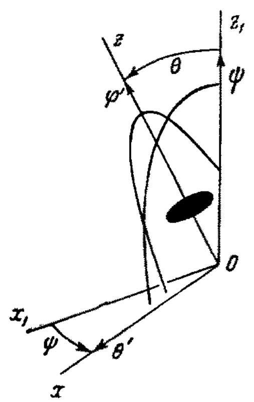

Fig. 1.

The projections of the angular velocity of the casing onto the x, y, and z axes are given by

$$p^\circ = \theta', \qquad q^\circ = \psi' \sin\theta, \qquad r^\circ = \psi' \cos\theta$$

The projections of the angular velocity of the gyroscope onto these same axes are

$$p = \theta', \qquad q = \psi' \sin\theta, \qquad r = \varphi' + \psi' \cos\theta$$

The kinetic energy of the external ring is

$$\frac{1}{2} J \psi'^2$$

where J is the moment of inertia of the ring about the z_1 axis.

The kinetic energy of the casing is

$$\frac{1}{2}(A^\circ p^{\circ 2} + B^\circ q^{\circ 2} + C^\circ r^2)$$

where A^0, B^0, and C^0 are the moments of inertia of the casing about the x, y, and z axes respectively. We shall assume these to be the principal axes of the ellipsoid of inertia of the gyroscope casing about the fixed point O.

The kinetic energy of the gyroscope is

$$\frac{1}{2}(Ap^2 + Aq^2 + Cr^2)$$

where A is the moment of inertia of the gyroscope about the x or y axis, and C is the moment of inertia about the z axis.

We shall assume the ellipsoid of inertia of the gyroscope about O to be an ellipsoid of rotation about the z axis.

Then the total kinetic energy is given by

$$2T = \psi'^2 (J + (A + B^\circ)\sin^2\theta + C^\circ \cos^2\theta) + (A + A^\circ)\theta'^2 + C(\varphi' + \psi' \cos\theta)^2$$

Using this expression we can write Lagrange's equations, since ψ, θ, and ϕ are independent variables which fully determine the position of the system (holonomic variables). These equations are

$$(A + A^\circ)\theta'' - \varphi'^2 (A - C + B^\circ - C^\circ)\sin\theta\cos\theta + C\varphi'\psi' \sin\theta = Q_\theta$$

$$\frac{d}{dt}\{\psi' (J + (A + B^\circ)\sin^2\theta + C^\circ \cos^2\theta) + C\cos\theta(\varphi' + \psi' \cos\theta)\} = Q_\psi$$

$$\frac{d}{dt} C(\varphi' - \psi' \cos\theta) = Q_\varphi$$

Here $Q_\theta \delta\theta$ is the work done by the forces(acting on the system) when the system rotates through an angle $\delta\theta$ about the x axis, $Q_\psi \delta\psi$ is the work done when the ring together with the casing and gyroscope rotates through an angle $\delta\psi$ about the z_1 axis, and $Q_\phi \delta\phi$ is the work done when the gyroscope rotates through an angle $\delta\phi$.

2. Let us assume that the suspension is frictionless, that the forces acting on the system are gravitational, that the center of gravity of the casing and gyroscope lies at a distance ζ from the origin O along the z axis, and that the z_1 axis is vertical.

Then Lagrznge's equations of motion lead to the following first integrals:

$$\varphi' + \psi' \cos\theta = r_0$$

$$\psi' (J + (A + B^\circ) \sin^2\theta + C^\circ \cos^2\theta) + C(\varphi' + \psi' \cos\theta) \cos\theta = k$$

$$\psi'^2 (J + (A + B^\circ) \sin^2\theta + C^\circ \cos^2\theta) + (A + A^\circ)\theta'^2 + C(\varphi' + \psi' \cos\theta)^2 = -2mg\zeta\cos\theta + h$$

The first two of these are the integrals related to the cyclic coordinates ϕ and ψ.

The last integral, representing the kinetic energy, could also have been obtained directly, since the actual displacements are possible displacements, and the force can be described by the potential

$$U = -mg\zeta\cos\theta$$

Here r_0, k, and h are the constant values of the first integrals, and m is the mass of the gyroscope and casing.

3. We can now write the differential equations for the Euler angles ψ, θ, and ϕ. These equations are

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{(\alpha - au)(\varepsilon - eu^2) - (\beta - br_0u)^2}{\varepsilon - eu^2}$$

$$\frac{d\psi}{dt} = \frac{\beta - br_0u}{\varepsilon - eu^2}$$

$$\frac{d\varphi}{dt} = r_0 - u\,\frac{\beta - br_0u}{\varepsilon - eu^2}$$

Here we have used the following notation

$$u = \cos\theta$$

$$\alpha = \frac{h - Cr_0^2}{A + A^\circ}, \qquad a = \frac{2mg\zeta}{A + A^\circ} > 0, \qquad \varepsilon = \frac{J + A + B^\circ}{A + A^\circ} > 0$$

$$e = \frac{A + B^\circ - C^\circ}{A + A^\circ}, \qquad \beta = \frac{k}{A + A^\circ}, \qquad b = \frac{C}{A + A^\circ} > 0$$

It is simplest to start by integrating the first equation, which gives

$$\int_{u_0}^{u} \frac{(\varepsilon - eu^2)\,du}{\sqrt{[(\alpha - au)(\varepsilon - eu^2) - (\beta - br_0u)^2](\varepsilon - eu^2)(1 - u^2)}} = t - t_0$$

After inverting this hyperelliptic integral (that is, after solving for u in terms of t), the calculation of ψ and ϕ reduces to quadratures.

Let us write

$$f(u) = (\alpha - au)(\varepsilon - eu^2) - (\beta - br_0u)^2$$

If $e > 0$, which is true for most practically interesting cases, the polynomial $f(u)$ has three real roots lying in the intervals

$$-\sqrt{\frac{\varepsilon}{e}} < u_1 \leqslant u_0, \qquad u_0 \leqslant u_2 < \sqrt{\frac{\varepsilon}{e}}, \qquad \sqrt{\frac{\varepsilon}{e}} < u' < \infty$$

The values of u in the mechanical problem vary within that interval between two adjacent points of the set $\{-1, +1, u_1, u_2\}$ which contains the point u_0.

The forms of the differential equations for Euler's angles and the definite sequence in which they are integrated show that in our problem the nutational motion of the gyroscope axis is a controlling factor, as it is in the case of Lagrange motion.

For the case of pseudoregular precession given by the initial conditions

$$\theta_0 \neq 0, \qquad p_0 = 0, \qquad q_0 = 0$$

and if r_0 is very large, we have

$$\beta - b r_0 u_0 = 0, \qquad \alpha - a u_0 = 0$$

and therefore

$$f(u) = (u_0 - u)\,[a(\varepsilon - eu^2) - b^2 r_0^2 (u_0 - u)]$$

This gives

$$u_0 - u_1 = \frac{a(\varepsilon - eu^2)}{b^2 r_0^2} > 0$$

so that for sufficiently large r_0 we find that u varies in the interval $(u_1, u_2 = u_0)$, and this interval gets smaller as r_0 increases.

In our problem regular precession is also possible. In this case the polynomial $f(u)$ has a multiple root $u_1 = u_2 = u_0$, the conditions for which are

$$f(u_0) = 0, \ f'(u_0) = 0$$

From these we obtain

$$(A + B^\circ - C^\circ)\, u_0 \psi_0'^2 - C r_0 \psi_0' + mg\zeta = 0$$

The condition that this quadratic equation for ψ_0' have real roots is

$$C^2 r_0^2 - 4(A + B^\circ - C^\circ)\, u_0\, mg\zeta > 0$$

or

$$C^2 \varphi_0'^2 - 4(A - C + B^\circ - C^\circ)\, u_0 mg\zeta > 0$$

The restriction to small deviations of u from unity, for the practically interesting cases $e > 0$, can be obtained in the same way as in the Lagrange case. For this it is sufficient to require that the roots of the polynomial

$$f(1 - \delta - z)$$

be negative, which means that the roots of $f(u)$ lie to the right of $1-\delta$. Here δ is a small positive constant. Such a δ will be the smallest positive number satisfying the inequalities

$$b^2 r_0^2 - 2ae + e(\alpha - a) + 3ae\delta > 0$$

$$\{b^2 r_0^2 + (\alpha - a(1-\delta)) - e - 2ae(1-\delta)\}\{2br_0(\beta - br_0(1-\delta)) - a(\varepsilon - e(1-\delta)^2) - \\ - 2e(1-\delta)(\alpha - a(1-\delta))\} - ae\{-(\alpha - a(1-\delta))(\varepsilon - e(1-\delta)^2) + (\beta - br_0(1-\delta))^2\} > 0$$

$$(\beta - br_0(1-\delta))^2 - (\alpha - a(1-\delta))(\varepsilon - e(1-\delta)^2) > 0$$

For the practically interesting case

$$\theta_0 = 0, \qquad \theta_0'^2 > 0, \ \psi_0' = 0$$

these equations can be analyzed in the same way as for Lagrange motion [4].

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